

# Where does $AC_\omega$ hide in constructive reals?

Exact real arithmetic in Cubical Agda (?)

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Dagstuhl Seminar 26121

16 March 2026

**Setting:** Homotopy Type Theory, formalized in Cubical Agda, plain (?) constructive foundations.

Standard good choice for constructive reals: Cauchy sequences with modulus of convergence, modulo  $\sim$ .

$$\mathbb{R}_C = \text{CauchySeq}/\sim$$

- Opaque elimination: we need a function on *all* sequences plus a proof it respects  $\sim$ .
- No structural cases: “rational or limit?”
- No representatives: given a class  $[s]$ , but you cannot pick one sequence.

HoTT/cubical approach: build reals as a higher inductive type.

$$\text{rat} : \mathbb{Q} \rightarrow \mathbb{R}_H, \quad \text{lim} : (\mathbb{Q}_+ \rightarrow \mathbb{R}_H) \rightarrow \text{Cauchy} \rightarrow \mathbb{R}_H, \quad \text{eq}_{\mathbb{R}_H}$$

- Structural elimination: case-split on  $\text{rat}$ ,  $\text{lim}$ ,  $\text{eq}_{\mathbb{R}_H}$ . For propositional targets, only  $\text{rat}$  and  $\text{lim}$  cases needed.
- Still hard: you cannot extract digits or computational content from a real.

- A coinductive stream is an infinite sequence:

$$\mathbf{3}^{\mathbb{N}} = \text{Stream}\{-1, 0, +1\}, \quad \text{stream} \rightarrow \mathbb{R}_H(s) = \sum_{i=0}^{\infty} \frac{s_i}{2^{i+1}} \in [-1, 1]$$

Example:  $+1 :: 0 :: -1 :: 0 :: \dots$  represents  $\frac{3}{8}$ .

- Motivation: exact real arithmetic and lazy computation.
- Problem: different streams can encode the same real, so we quotient:  
 $\mathbb{R}_{SD} = (\mathbb{Z} \times \mathbf{3}^{\mathbb{N}}) / \approx_{\text{ext}}.$

What can we say about the relation between  $\mathbb{R}_{SD}$  and  $\mathbb{R}_H$ ?

# The $AC_\omega$ Step

Is  $\mathbb{R}_{SD}$  to be Cauchy-complete: does every Cauchy family  $f : \mathbb{Q}_+ \rightarrow \mathbb{R}_{SD}$  have a limit?

The limit constructor needs *actual streams*, but  $f(\delta)$  is a quotient class — we know a representative  $(k, s)$  is inside, but the quotient hides it.

$$\underbrace{\forall \delta. \|\text{fiber } [-] (f \delta)\|_1}_{\text{for each } \delta, \text{ a rep merely exists}} \xrightarrow{AC_\omega} \underbrace{\|\forall \delta. \text{fiber } [-] (f \delta)\|_1}_{\text{a simultaneous choice for all } \delta}$$

We have injective  $\mathbb{R}_{\text{SD}} \rightarrow \mathbb{R}_{\text{H}}$ . For the other direction: given  $x : \mathbb{R}_{\text{H}}$ , can we build a preimage in  $\mathbb{R}_{\text{SD}}$ ?

Booij (2021): attach an *approximation oracle* as extra structure.

$$\text{LocatorA}(x) = (\varepsilon : \mathbb{Q}_+) \rightarrow \Sigma(q : \mathbb{Q}). x \sim_{[\varepsilon]} \text{rat}(q)$$

Different locators for the same real give different rational approximations.

We can build locators constructively for each constructor of  $\mathbb{R}_H$ :

- `ratLocator`:  $q$  is its own approximation at any  $\varepsilon$ .
- `streamLocator`: truncate a stream at position  $n$  to get a rational partial sum.
- `limLocator`: given locators for each approximant, pick one close enough and use its locator.

Given `LocatorA( $x$ )`, we can constructively extract a signed-digit representative in  $\mathbb{R}_{SD}$ . No choice needed.

# Locators consequences

Does every real come with a locator?

$$\text{AllLoc}_{\mathbb{R}_H} = (x : \mathbb{R}_H) \rightarrow \text{LocatorA}(x)$$

This is non-constructive: assembling pointwise truncated locators into a coherent family is  $\text{AC}_\omega$ .

$$\text{AC}_\omega \xrightarrow{\text{standard}} \text{AllLoc}_{\mathbb{R}_H} \xrightarrow{?} \mathbb{R}_{\text{SD}} \simeq \mathbb{R}_H$$

- $\text{AC}_\omega \rightarrow \text{AllLoc}_{\mathbb{R}_H}$ : via  $\text{Elim}_{\mathbb{R}_H}\text{-Prop}$  + countable choice.
- $\text{AllLoc}_{\mathbb{R}_H} \rightarrow \mathbb{R}_{\text{SD}} \simeq \mathbb{R}_H$ : done
- $\mathbb{R}_{\text{SD}} \simeq \mathbb{R}_H \rightarrow \text{AllLoc}_{\mathbb{R}_H}$  ?

# The Gap (1): what works

Assume  $\mathbb{R}_{\text{SD}} \simeq \mathbb{R}_{\text{H}}$ . Can we get locators for all reals?

$\mathbb{R}_{\text{SD}} \rightarrow \mathbb{R}_{\text{H}}$  is injective, so each fiber has at most one element (propositional).

Truncation of a proposition is harmless:

$$\|\text{fiber}(\text{to}\mathbb{R}_{\text{H}}, x)\|_1 \rightarrow \text{fiber}(\text{to}\mathbb{R}_{\text{H}}, x)$$

From surjectivity we get an actual section  $\mathbb{R}_{\text{H}} \rightarrow \mathbb{R}_{\text{SD}}$ . No choice needed.

## The Gap (2): what breaks

We have  $y : \mathbb{R}_{SD}$  with  $\text{to}\mathbb{R}_H(y) = x$ . We want  $\text{LocatorA}(x)$ .

$y$  is a quotient class  $[r]$ .

The obvious map  $r \mapsto \text{streamLocator}(r)$  is not well-defined on the quotient: equivalent reps give different locators, and  $\text{LocatorA}$  is not propositional.

Getting a preimage  $\neq$  extracting computational structure from it.

- Back to Bishop: use setoids instead of quotients, you always have the underlying Cauchy sequence but equality is external. Setoid hell.
- In the HIT/quotient formalization, to define  $f : \mathbb{R}_{\text{SD}} \rightarrow \text{LocatorA}(\text{to}\mathbb{R}_{\text{H}}(-))$ , the quotient eliminator demands:

$$r \approx_{\text{ext}} r' \implies f(r) = f(r')$$

- But  $\text{streamLocator}(r)$  and  $\text{streamLocator}(r')$  return *different* rational approximations at each  $\varepsilon$ . Since  $\text{LocatorA}$  is a set (not propositional), these are genuinely unequal.

Punchline: “bijection” and “uniform computability” may genuinely separate in HIT-based constructive analysis.

# Open Questions

- Is  $\text{AllLoc}_{\mathbb{R}_H}$  strictly stronger than  $\mathbb{R}_{SD} \simeq \mathbb{R}_H$ ?
- Can the coherence gap be closed by a canonical locator construction?
- Does Booi's locator story transfer from setoid Cauchy reals to the HIT?
- Where else does “bijection  $\neq$  uniform computability” arise in HIT formalizations?
- Higher coinductive interval types? It could build signed-digit structure and path equality simultaneously
- Coinductive locators?

## Constructive analysis:

- Bishop (1967), *Foundations of Constructive Analysis*
- Bishop–Bridges (1985), *Constructive Analysis*

## Signed-digit / exact real arithmetic:

- Boehm et al. (1986), *Exact Real Arithmetic: A Case Study in Higher Order Programming*
- Berger–Seisenberger (2012), *From Coinductive Proofs to Exact Real Arithmetic*
- Schwichtenberg–Wiesnet (2021), *Limits in the Signed Digit Representation*

## HoTT reals and locators:

- HoTT Book (2013), Ch. 11 — Cauchy reals as a HIT
- Bauer (2016), *The Real Numbers in Homotopy Type Theory* (CCA slides)
- Booiij (2020), *The HoTT Reals are the Cauchy Completion of the Rationals*
- Booiij (2021), *Extensional Constructive Real Analysis via Locators*, MSCS
- Escardó–Simpson (2002), *A Universal Characterization of the Closed Real Interval*, TCS