

An Agda Implementation of the Modal Logic S4.2: First Investigations

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The Modal Logic S4.2

S4.2 extends S4 with the Geach axiom: $\Diamond\Box\phi \rightarrow \Box\Diamond\phi$.

Semantically characterized by reflexive, transitive, and confluent Kripke frames.

Applications:

- Physics: Diodorean interpretation of time in Minkowski space
- Set theory: Modal logic of forcing in ZFC
- Algebra: Modal logic of abelian groups
- Knowledge representation: Foundation for epistemic logic

Why Study and Mechanize S4.2?

- Robust proof systems with strong structural properties are essential for effective reasoning
- Mechanization provides formal verification, clear foundational understanding, and potential for extensions

Goal

- 1) Develop a sequent calculus for S4.2
- 2) Mechanize it
- 3) Provide and implement syntactical proof of cut-elimination

Step 1: defining the sequent calculus $E_{S4.2}$

Key features:

- Propositional language with modalities $\Box$ and $\Diamond$
- Formulas annotated with *positions*: A^s where s is a finite set of tokens
- Positions manage modal information without explicit accessibility relations
- Only rules for modal operators change formula positions

Step 1: defining the sequent calculus $E_{S4.2}$

$E_{S4.2}$ Modal Rules

$$\frac{\Gamma, A^{s,t} \vdash \Delta}{\Gamma, \Box A^s \vdash \Delta} \Box \vdash$$

$$\frac{\Gamma \vdash A^{s,x}, \Delta}{\Gamma \vdash \Box A^s, \Delta} \vdash \Box$$

$$\frac{\Gamma, A^{s,x} \vdash \Delta}{\Gamma, \Diamond A^s \vdash \Delta} \Diamond \vdash$$

$$\frac{\Gamma \vdash A^{s,t}, \Delta}{\Gamma \vdash \Diamond A^s, \Delta} \vdash \Diamond$$

Modal rules employ eigentokens: $x \notin s$ and x fresh for Γ, Δ for $\vdash \Box$ and $\Diamond \vdash$

Step 2: Mechanizing $E_{S4.2}$

Why Agda?

- Functional "proof term" approach vs imperative "proof scripts"
- Clean Unicode syntax close to mathematical notation
- Based on constructive type theory, suited for developing intuitionistic extensions

Step 2: Mechanizing $E_{S4.2}$

Deriving the Geach Axiom

$$\frac{\frac{\frac{A^{x,y} \vdash A^{x,y}}{\Box A^x \vdash A^{x,y}} (\Box \vdash) \quad \frac{\Box A^x \vdash A^{x,y}}{\Box A^x \vdash \Diamond A^y} (\vdash \Diamond)}{\Box A^x \vdash \Box \Diamond A^\emptyset} (\vdash \Box) \quad \frac{\Box A^x \vdash \Box \Diamond A^\emptyset}{\Diamond \Box A^\emptyset \vdash \Box \Diamond A^\emptyset} (\Diamond \vdash)}{\vdash (\Diamond \Box A \rightarrow \Box \Diamond A)^\emptyset} (\vdash \rightarrow)$$

Step 2: Mechanizing $E_{S4.2}$

Implementing the derivation

```
geachAxiom :  $\forall \{n\} A$ 
   $\rightarrow$  (Proof {suc (suc n)} ( $\Box \vdash [((\Diamond (\Box A)) \Rightarrow (\Box (\Diamond A)) \wedge \perp]$ )))
geachAxiom {A = A} =
   $\vdash \Rightarrow$ 
  ( $\Diamond \vdash \{x = x\} \{ \Gamma = \Box \} \notin \perp (\lambda ()) (\lambda ())$ 
  ( $\vdash \Box \{x = y\} \notin \perp$  (y-fresh-helper {A = A})  $\notin \perp$ 
  ( $\vdash \Diamond \{s = \perp \cup \{y\}\} \{t = \{x\}\}$ 
  ( $\Box \vdash \{A = A\} \{s = \perp \cup \{x\}\} \{t = \{y\}\} \{ \Gamma = \Box \}$ 
  Ax))))
```

Step 3: the Cut-Elimination Theorem

Theorem (Cut-Elimination)

If Π is a proof of $\Gamma \vdash \Delta$, then there exists a cut-free proof Π^ of $\Gamma \vdash \Delta$.*

Idea of the pen-on-paper proof:

- Gentzen techniques adapted for modal position-based sequent calculi
- Induction on $\langle \delta[\Pi], h(\Pi) \rangle$
- Uses a Mix Lemma to eliminate cuts
- Termination checking with δ complexity measure still in progress

Corollaries

- Subformula Property
- (Syntactical) proof of Consistency

Future Work

Ongoing developments:

- Complete proofs of auxiliary lemmas in Agda
- Implement the Weak Completeness theorem
- Formalize semantics based on semilattices with minimum and the Soundness Theorem

Long-term goals:

- Mechanize semantics for S4.2 as modal logic of abelian groups
- Investigate intuitionistic version of S4.2
- Investigate Curry-Howard correspondence and type systems

Part of the research project NAMOR (New Agda MOdal Realization), an extensible framework for normal and non-normal modal logics.

References I

Thank You!

Questions?

Identity rules

$$\frac{}{A^s \vdash A^s} Ax \qquad \frac{\Gamma_1 \vdash A^s, \Delta_1 \quad \Gamma_2, A^s \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} Cut$$

Structural rules

$$\begin{array}{c} \frac{\Gamma \vdash \Delta}{\Gamma, A^s \vdash \Delta} W \vdash \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A^s, \Delta} \vdash W \\[10pt] \frac{\Gamma, A^s, A^s \vdash \Delta}{\Gamma, A^s \vdash \Delta} C \vdash \qquad \frac{\Gamma \vdash A^s, A^s, \Delta}{\Gamma \vdash A^s, \Delta} \vdash C \\[10pt] \frac{\Gamma_1, A^s, B^t, \Gamma_2 \vdash \Delta}{\Gamma_1, B^t, A^s, \Gamma_2 \vdash \Delta} Exc \vdash \qquad \frac{\Gamma \vdash \Delta_1, A^s, B^t, \Delta_2}{\Gamma \vdash \Delta_1, B^t, A^s, \Delta_2} \vdash Exc \end{array}$$

Propositional rules

$$\begin{array}{c}
 \frac{\Gamma, A^s \vdash \Delta}{\Gamma, A \wedge B^s \vdash \Delta} \wedge_1 \vdash \quad \frac{\Gamma, B^s \vdash \Delta}{\Gamma, A \wedge B^s \vdash \Delta} \wedge_2 \vdash \quad \frac{\Gamma_1 \vdash A^s, \Delta_1 \quad \Gamma_2 \vdash B^s, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \wedge B^s, \Delta_1, \Delta_2} \vdash \wedge \\
 \\
 \frac{\Gamma_1, A^s \vdash \Delta_1 \quad \Gamma_2, B^s \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \vee B^s \vdash \Delta_1, \Delta_2} \vee \vdash \quad \frac{\Gamma \vdash A^s, \Delta}{\Gamma \vdash A \vee B^s, \Delta} \vdash \vee_1 \quad \frac{\Gamma \vdash B^s, \Delta}{\Gamma \vdash A \vee B^s, \Delta} \vdash \vee_2 \\
 \\
 \frac{\Gamma_1 \vdash A^s, \Delta_1 \quad \Gamma_2, B^s \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \rightarrow B^s \vdash \Delta_1, \Delta_2} \rightarrow \vdash \quad \frac{\Gamma, A^s \vdash B^s, \Delta}{\Gamma \vdash A \rightarrow B^s, \Delta} \vdash \rightarrow
 \end{array}$$

Modal rules

$$\frac{\Gamma, A^{s,t} \vdash \Delta}{\Gamma, \Box A^s \vdash \Delta} \Box \vdash \quad \frac{\Gamma \vdash A^{s,x}, \Delta}{\Gamma \vdash \Box A^s, \Delta} \vdash \Box$$

$$\frac{\Gamma, A^{s,x} \vdash \Delta}{\Gamma, \Diamond A^s \vdash \Delta} \Diamond \vdash \quad \frac{\Gamma \vdash A^{s,t}, \Delta}{\Gamma \vdash \Diamond A^s, \Delta} \vdash \Diamond$$

with the constraints: $x \notin s$ and x fresh for Γ, Δ for $\vdash \Box$ and $\Diamond \vdash$.

Modal Rules in Agda

$\Box \vdash : \forall \{n\} A \{s\} t : \text{position } \{n\} \{ \Gamma \Delta : \text{List } (\text{pf } \{n\}) \}$
 $\rightarrow \text{Proof } (\Gamma , [(A \hat{\wedge} s \cup t)] \vdash \Delta)$
 $\rightarrow \text{Proof } (\Gamma , [(\Box A \hat{\wedge} s)] \vdash \Delta)$

$\vdash \Box : \forall \{n\} A \{x\} : \text{token } \{n\} \{s\} : \text{position } \{n\} \{ \Gamma \Delta : \text{List } (\text{pf } \{n\}) \}$
 $\rightarrow x \notin s$
 $\rightarrow \text{fresh } x \Gamma$
 $\rightarrow \text{fresh } x \Delta$
 $\rightarrow \text{Proof } (\Gamma \vdash [(A \hat{\wedge} s \cup \{x\})] , \Delta)$
 $\rightarrow \text{Proof } (\Gamma \vdash [(\Box A \hat{\wedge} s)] , \Delta)$

Modal Rules in Agda

$\Diamond \vdash : \forall \{n\} A \{x : \text{token } \{n\}\} \{s : \text{position } \{n\}\} \{\Gamma \Delta : \text{List } (\text{pf } \{n\})\}$
 $\rightarrow x \notin s$
 $\rightarrow \text{fresh } x \Gamma$
 $\rightarrow \text{fresh } x \Delta$
 $\rightarrow \text{Proof } (\Gamma , [(A \hat{\wedge} s \cup \{x\})] \vdash \Delta)$
 $\rightarrow \text{Proof } (\Gamma , [(\Diamond A \hat{\wedge} s)] \vdash \Delta)$

$\vdash \Diamond : \forall \{n\} A \{s\} t : \text{position } \{n\} \{\Gamma \Delta : \text{List } (\text{pf } \{n\})\}$
 $\rightarrow \text{Proof } (\Gamma \vdash [(A \hat{\wedge} s \cup t)] , \Delta)$
 $\rightarrow \text{Proof } (\Gamma \vdash [(\Diamond A \hat{\wedge} s)] , \Delta)$