

NAMOR: a New Agda Library for Modal Extended Sequents

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Riccardo Borsetto Margherita Zorzi

University of Verona

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Modal logic extends logic with modalities

- Used to reason about necessity ($\Box$), possibility ($\Diamond$), knowledge, belief, time, etc.
- Applications in mathematics, physics, symbolic AI, knowledge representation and epistemic properties in multi-agent distributed systems

Various deductive systems for modal logic: labeled systems, 2-sequents, linear sequents, nested sequents, hypersequents etc.

Position-based extended sequent calculus

Desired metatheoretical properties

- Normalisation/Cut-elimination
- Subformula property
- Consistency

Position-based extended sequent calculus

- Formulas annotated with **positions**: A^α where α is a sequence or a set of tokens
- Positions manage modal information without explicit accessibility relations
- Same rules for different systems obtained by tuning constraints
- Direct syntactical proof of the cut-elimination theorem

What is Agda?

- A dependently typed functional programming language and proof assistant
- **Curry-Howard correspondence**: propositions $\leftrightarrow$ types, proofs $\leftrightarrow$ programs
- Used for formal verification, theorem proving, and verified software development

Why Agda?

- Expressive theory that allows deep embeddings of complex systems
- Functional *proof term* approach vs imperative *proof scripts*
- Clean Unicode syntax close to mathematical notation

NAMOR (New Agda MOdal Realization)

Agda position-based framework for normal extensions of modal logic K, from K to S5

Example (Modal rules)

$$\frac{\Gamma, A^\beta \vdash \Delta}{\Gamma, \Box A^\alpha \vdash \Delta} \Box \vdash \quad \frac{\Gamma \vdash A^{\alpha \circ \langle x \rangle}, \Delta}{\Gamma \vdash \Box A^\alpha, \Delta} \vdash \Box$$
$$\frac{\Gamma, A^{\alpha \circ \langle x \rangle} \vdash \Delta}{\Gamma, \Diamond A^\alpha \vdash \Delta} \Diamond \vdash \quad \frac{\Gamma \vdash A^\beta, \Delta}{\Gamma \vdash \Diamond A^\alpha, \Delta} \vdash \Diamond$$

$\vdash \Box$ and $\Diamond \vdash$ employ eigentokens with the constraint: no position in Γ, Δ starts with $\alpha \circ \langle x \rangle$.

Constraints on modal rules

Calculus	Constraints on the rules $\Box \vdash$ and $\vdash \Diamond$
S5	no constraints
S4.2	$\alpha \subseteq \beta$ (subset)
S4	$\alpha \sqsubseteq \beta$ (prefix)
T	$\alpha \triangleleft^0 \beta$ (predecessor or equal)
D	$\alpha \triangleleft \beta$ (predecessor)
D4	$\alpha \sqsubset \beta$ (proper prefix)
K4	$\alpha \sqsubset \beta$ (proper prefix) and there exists $B^{\delta \circ \beta \circ \eta}$ in Γ, Δ
K	$\alpha \triangleleft \beta$ (predecessor) and there exists $B^{\delta \circ \beta \circ \eta}$ in Γ, Δ

Implementation of modal constraints in Agda

```
modalConstraint : Logic → position → position → List pf → List pf → Type
modalConstraint S5  $\alpha$   $\beta$   $\Gamma$   $\Delta$  = Unit
modalConstraint S4dot2  $\alpha$   $\beta$   $\Gamma$   $\Delta$  =  $\alpha \sqsubseteq \beta$ 
modalConstraint S4  $\alpha$   $\beta$   $\Gamma$   $\Delta$  =  $\alpha \sqsubseteq \beta$ 
modalConstraint T  $\alpha$   $\beta$   $\Gamma$   $\Delta$  =  $\alpha \triangleleft^0 \beta$ 
modalConstraint D  $\alpha$   $\beta$   $\Gamma$   $\Delta$  =  $\alpha \triangleleft \beta$ 
modalConstraint D4  $\alpha$   $\beta$   $\Gamma$   $\Delta$  =  $\alpha \sqsubseteq \beta$ 
modalConstraint K4  $\alpha$   $\beta$   $\Gamma$   $\Delta$  =  $\alpha \sqsubseteq \beta \times (\Gamma \text{ ,, } \Delta) \text{ has } \beta$ 
modalConstraint K  $\alpha$   $\beta$   $\Gamma$   $\Delta$  =  $\alpha \triangleleft \beta \times (\Gamma \text{ ,, } \Delta) \text{ has } \beta$ 
```

Example: the Geach axiom

The Geach axiom $\Diamond\Box A \rightarrow \Box\Diamond A$ is the characteristic axiom of S4.2, intermediate logic between S4 and S5, commonly regarded as the correct logic for epistemic representation of knowledge.

Theorem (Geach axiom)

$$\frac{\frac{\frac{\frac{\frac{A^{x,y} \vdash A^{x,y}}{\Box A^x \vdash A^{x,y}} \vdash \Diamond}{\Box A^x \vdash \Diamond A^y} \vdash \Box}{\Box A^x \vdash \Box \Diamond A^\emptyset} \vdash \Diamond}{\Diamond \Box A^\emptyset \vdash \Box \Diamond A^\emptyset} \vdash \rightarrow}{\vdash (\Diamond \Box A \rightarrow \Box \Diamond A)^\emptyset}$$

Challenge: S4-derivation of the Geach axiom in Agda

Example

```
axiomG-S4-FAILED :  $\forall \{A\} \rightarrow (\text{Proof } ([] \vdash [((\Diamond (\Box A)) \Rightarrow (\Box (\Diamond A)) \wedge [])]))$ 
axiomG-S4-FAILED =
   $\vdash \Rightarrow$ 
  ( $\Diamond \vdash \{x = \text{"x"}\} \{\Gamma = []\} (\lambda \{ (\text{here } ()) ; (\text{there } ()) \}$ 
  ( $\vdash \Box \{x = \text{"y"}\} (\lambda \{ (\text{there } (\text{here } ())) ; (\text{there } (\text{there } ())) \}$ 
  ( $\vdash \Diamond \{M = \text{S4}\} \{\alpha = \langle \text{"y"} \rangle\} \{\beta = \langle \text{"x"} \rangle \circ \langle \text{"y"} \rangle\} \{! \ !\} \{! \ !\} )) -- \text{FAILURE}$ 
```

S4.2-derivation of the Geach axiom in Agda

Geach axiom derivation

```
axiomG : ∀ {A} → (Proof ([] ⊢ [((◇ (□ A)) ⇒ (□ (◇ A)) ^ [])]))
axiomG =
  ⊢⇒
  (◇⊢ {x = "x"} {Γ = []} (λ { (here ()) ; (there ()) } )
  (⊢□ {x = "y"} (λ { (there (here ())) ; (there (there ())) } )
  (⊢◇ {M = S4dot2} {α = ⟨"y"⟩} {β = ⟨"x"⟩ ∘ ⟨"y"⟩} (λ {x} → there)
  (□⊢ {M = S4dot2} {α = ⟨"x"⟩} {β = ⟨"x"⟩ ∘ ⟨"y"⟩} {Γ = []} (xs⊆xs++ys ⟨ "x" ⟩ ⟨ "y" ⟩ )
  Ax ))))
```

Example motivations

- Modular constraint-based approach showcase
- Type safety prevents invalid derivations

Cut-elimination	Implement the cut-elimination proof once for all logics
Weak completeness	Give a weak completeness proof w.r.t. to the axiom system
Soundness	Complete the semantic framework
Non-normal logics	Extend the library beyond normal modal logics

References

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- [Neg11] Sara Negri. “Proof Theory for Modal Logic”. In: *Philosophy Compass* 6.8 (2011), pp. 523–538.
- [Sim94] Alex K. Simpson. “The Proof Theory and Semantics of Intuitionistic Modal Logic”. In: (July 1994).
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Thank you for your attention

NAMOR framework: axiom, cut and structural rules

$$\frac{}{A^\alpha \vdash A^\alpha} Ax \qquad \frac{\Gamma_1 \vdash A^\alpha, \Delta_1 \quad \Gamma_2, A^\alpha \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} Cut$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A^\alpha \vdash \Delta} W \vdash \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A^\alpha, \Delta} \vdash W$$

$$\frac{\Gamma, A^\alpha, A^\alpha \vdash \Delta}{\Gamma, A^\alpha \vdash \Delta} C \vdash \qquad \frac{\Gamma \vdash A^\alpha, A^\alpha, \Delta}{\Gamma \vdash A^\alpha, \Delta} \vdash C$$

$$\frac{\Gamma_1, A^\alpha, B^\beta, \Gamma_2 \vdash \Delta}{\Gamma_1, B^\beta, A^\alpha, \Gamma_2 \vdash \Delta} Exc \vdash \qquad \frac{\Gamma \vdash \Delta_1, A^\alpha, B^\beta, \Delta_2}{\Gamma \vdash \Delta_1, B^\beta, A^\alpha, \Delta_2} \vdash Exc$$

NAMOR framework: propositional rules

$$\begin{array}{c} \frac{\Gamma, A^\alpha \vdash \Delta}{\Gamma, A \wedge B^\alpha \vdash \Delta} \wedge_1 \vdash \quad \frac{\Gamma, B^\alpha \vdash \Delta}{\Gamma, A \wedge B^\alpha \vdash \Delta} \wedge_2 \vdash \quad \frac{\Gamma_1 \vdash A^\alpha, \Delta_1 \quad \Gamma_2 \vdash B^\alpha, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \wedge B^\alpha, \Delta_1, \Delta_2} \vdash \wedge \\[2ex] \frac{\Gamma_1, A^\alpha \vdash \Delta_1 \quad \Gamma_2, B^\alpha \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \vee B^\alpha \vdash \Delta_1, \Delta_2} \vee \vdash \quad \frac{\Gamma \vdash A^\alpha, \Delta}{\Gamma \vdash A \vee B^\alpha, \Delta} \vdash \vee_1 \quad \frac{\Gamma \vdash B^\alpha, \Delta}{\Gamma \vdash A \vee B^\alpha, \Delta} \vdash \vee_2 \\[2ex] \frac{\Gamma_1 \vdash A^\alpha, \Delta_1 \quad \Gamma_2, B^\alpha \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \rightarrow B^\alpha \vdash \Delta_1, \Delta_2} \rightarrow \vdash \quad \frac{\Gamma, A^\alpha \vdash B^\alpha, \Delta}{\Gamma \vdash A \rightarrow B^\alpha, \Delta} \vdash \rightarrow \end{array}$$

NAMOR framework: modal rules

$$\frac{\Gamma, A^\beta \vdash \Delta}{\Gamma, \Box A^\alpha \vdash \Delta} \Box \vdash \quad \frac{\Gamma \vdash A^{\alpha \circ \langle x \rangle}, \Delta}{\Gamma \vdash \Box A^\alpha, \Delta} \vdash \Box$$
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$\vdash \Box$ and $\Diamond \vdash$ employ eigentokens with the constraint: no position in Γ, Δ starts with $\alpha \circ \langle x \rangle$.

Constraints on rules

Calculus	Constraints on the rules $\Box \vdash$ and $\vdash \Diamond$
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K	$\alpha \triangleleft \beta$ (predecessor) and there exists $B^{\delta \circ \beta \circ \eta}$ in Γ, Δ

For K and K4 we also have the following constraint on the cut rule:

$$\alpha \in \mathfrak{Init}[\Gamma_1, \Delta_1 - A^\alpha] \quad \text{or} \quad \alpha \in \mathfrak{Init}[\Gamma_2 - A^\alpha, \Delta_2]$$

NAMOR framework: rules in Agda

$Ax : \forall \{A \ \alpha\} \rightarrow \text{Proof} \ ([A \wedge \alpha] \vdash [A \wedge \alpha])$

$Cut : \forall \{M \ A \ \alpha \ \Gamma_1 \ \Gamma_2 \ \Delta_1 \ \Delta_2\}$
 $\rightarrow \text{cutConstraint} \ M \ A \ \alpha \ \Gamma_1 \ \Gamma_2 \ \Delta_1 \ \Delta_2$
 $\rightarrow \text{Proof} \ (\Gamma_1 \vdash [A \wedge \alpha] \text{ ,, } \Delta_1)$
 $\rightarrow \text{Proof} \ (\Gamma_2 \text{ ,, } [A \wedge \alpha] \vdash \Delta_2)$
 $\rightarrow \text{Proof} \ (\Gamma_1 \text{ ,, } \Gamma_2 \vdash \Delta_1 \text{ ,, } \Delta_2)$

NAMOR framework: rules in Agda

$W\vdash : \forall \{A \alpha \Gamma \Delta\}$
 $\rightarrow \text{Proof } (\Gamma \vdash \Delta)$
 $\rightarrow \text{Proof } (\Gamma ,, [A \wedge \alpha] \vdash \Delta)$

$\vdash W : \forall \{A \alpha \Gamma \Delta\}$
 $\rightarrow \text{Proof } (\Gamma \vdash \Delta)$
 $\rightarrow \text{Proof } (\Gamma \vdash [A \wedge \alpha] ,, \Delta)$

NAMOR framework: rules in Agda

$\text{C} \vdash : \forall \{A \alpha \Gamma \Delta\}$
→ $\text{Proof } (\Gamma \text{ ,, } [A \hat{\alpha}] \text{ ,, } [A \hat{\alpha}] \vdash \Delta)$
→ $\text{Proof } (\Gamma \text{ ,, } [A \hat{\alpha}] \vdash \Delta)$

$\vdash \text{C} : \forall \{A \alpha \Gamma \Delta\}$
→ $\text{Proof } (\Gamma \vdash [A \hat{\alpha}] \text{ ,, } [A \hat{\alpha}] \text{ ,, } \Delta)$
→ $\text{Proof } (\Gamma \vdash [A \hat{\alpha}] \text{ ,, } \Delta)$

NAMOR framework: rules in Agda

$\text{Exc} \vdash : \forall \{A\ B\ \beta\ \alpha\ \Gamma_1\ \Gamma_2\ \Delta\}$
 $\rightarrow \text{Proof}\ (\Gamma_1 \text{ ,, } [A \hat{\wedge} \alpha] \text{ ,, } [B \hat{\wedge} \beta] \text{ ,, } \Gamma_2 \vdash \Delta)$
 $\rightarrow \text{Proof}\ (\Gamma_1 \text{ ,, } [B \hat{\wedge} \beta] \text{ ,, } [A \hat{\wedge} \alpha] \text{ ,, } \Gamma_2 \vdash \Delta)$

$\vdash \text{Exc} : \forall \{A\ B\ \beta\ \alpha\ \Gamma\ \Delta_1\ \Delta_2\}$
 $\rightarrow \text{Proof}\ (\Gamma \vdash \Delta_1 \text{ ,, } [A \hat{\wedge} \alpha] \text{ ,, } [B \hat{\wedge} \beta] \text{ ,, } \Delta_2)$
 $\rightarrow \text{Proof}\ (\Gamma \vdash \Delta_1 \text{ ,, } [B \hat{\wedge} \beta] \text{ ,, } [A \hat{\wedge} \alpha] \text{ ,, } \Delta_2)$

NAMOR framework: rules in Agda

$\neg\vdash : \forall \{A \alpha \Gamma \Delta\}$
→ **Proof** ($\Gamma \vdash [A \wedge \alpha] \text{,,} \Delta$)
→ **Proof** ($\Gamma \text{,,} [\neg A \wedge \alpha] \vdash \Delta$)

$\vdash\neg : \forall \{A \alpha \Gamma \Delta\}$
→ **Proof** ($\Gamma \text{,,} [A \wedge \alpha] \vdash \Delta$)
→ **Proof** ($\Gamma \vdash [\neg A \wedge \alpha] \text{,,} \Delta$)

NAMOR framework: rules in Agda

$$\begin{aligned}\wedge_1 \vdash &: \forall \{A B \alpha \Gamma \Delta\} \\ &\rightarrow \text{Proof } (\Gamma \text{ ,, } [A \hat{\wedge} \alpha] \vdash \Delta) \\ &\rightarrow \text{Proof } (\Gamma \text{ ,, } [A \wedge B \hat{\wedge} \alpha] \vdash \Delta)\end{aligned}$$
$$\begin{aligned}\wedge_2 \vdash &: \forall \{A B \alpha \Gamma \Delta\} \\ &\rightarrow \text{Proof } (\Gamma \text{ ,, } [B \hat{\wedge} \alpha] \vdash \Delta) \\ &\rightarrow \text{Proof } (\Gamma \text{ ,, } [A \wedge B \hat{\wedge} \alpha] \vdash \Delta)\end{aligned}$$
$$\begin{aligned}\vdash \wedge &: \forall \{A B \alpha \Gamma_1 \Gamma_2 \Delta_1 \Delta_2\} \\ &\rightarrow \text{Proof } (\Gamma_1 \vdash [A \hat{\wedge} \alpha] \text{ ,, } \Delta_1) \\ &\rightarrow \text{Proof } (\Gamma_2 \vdash [B \hat{\wedge} \alpha] \text{ ,, } \Delta_2) \\ &\rightarrow \text{Proof } (\Gamma_1 \text{ ,, } \Gamma_2 \vdash [A \wedge B \hat{\wedge} \alpha] \text{ ,, } \Delta_1 \text{ ,, } \Delta_2)\end{aligned}$$

NAMOR framework: rules in Agda

$$\begin{aligned} \text{v}\vdash &: \forall \{A\ B\ \alpha\ \Gamma_1\ \Gamma_2\ \Delta_1\ \Delta_2\} \\ &\rightarrow \text{Proof}\ (\Gamma_1 \text{ ,, } [A \hat{\ } \alpha] \vdash \Delta_1) \\ &\rightarrow \text{Proof}\ (\Gamma_2 \text{ ,, } [B \hat{\ } \alpha] \vdash \Delta_2) \\ &\rightarrow \text{Proof}\ (\Gamma_1 \text{ ,, } \Gamma_2 \text{ ,, } [A \vee B \hat{\ } \alpha] \vdash \Delta_1 \text{ ,, } \Delta_2) \end{aligned}$$
$$\begin{aligned} \vdash V_1 &: \forall \{A\ B\ \alpha\ \Gamma\ \Delta\} \\ &\rightarrow \text{Proof}\ (\Gamma \vdash [A \hat{\ } \alpha] \text{ ,, } \Delta) \\ &\rightarrow \text{Proof}\ (\Gamma \vdash [A \vee B \hat{\ } \alpha] \text{ ,, } \Delta) \end{aligned}$$
$$\begin{aligned} \vdash V_2 &: \forall \{A\ B\ \alpha\ \Gamma\ \Delta\} \\ &\rightarrow \text{Proof}\ (\Gamma \vdash [B \hat{\ } \alpha] \text{ ,, } \Delta) \\ &\rightarrow \text{Proof}\ (\Gamma \vdash [A \vee B \hat{\ } \alpha] \text{ ,, } \Delta) \end{aligned}$$

NAMOR framework: rules in Agda

$$\begin{aligned} \Rightarrow \vdash & : \forall \{A \ B \ \alpha \ \Gamma_1 \ \Gamma_2 \ \Delta_1 \ \Delta_2\} \\ & \rightarrow \text{Proof} (\Gamma_1 \text{ ,, } [B \hat{=} \alpha] \vdash \Delta_1) \\ & \rightarrow \text{Proof} (\Gamma_2 \vdash [A \hat{=} \alpha] \text{ ,, } \Delta_2) \\ & \rightarrow \text{Proof} (\Gamma_1 \text{ ,, } \Gamma_2 \text{ ,, } [A \Rightarrow B \hat{=} \alpha] \vdash \Delta_1 \text{ ,, } \Delta_2) \end{aligned}$$
$$\begin{aligned} \vdash \Rightarrow & : \forall \{A \ B \ \alpha \ \Gamma \ \Delta\} \\ & \rightarrow \text{Proof} (\Gamma \text{ ,, } [A \hat{=} \alpha] \vdash [B \hat{=} \alpha] \text{ ,, } \Delta) \\ & \rightarrow \text{Proof} (\Gamma \vdash [A \Rightarrow B \hat{=} \alpha] \text{ ,, } \Delta) \end{aligned}$$

NAMOR framework: rules in Agda

$\Box \vdash : \forall \{M \ A \ \alpha \ \beta \ \Gamma \ \Delta\}$
→ modalConstraint $M \ \alpha \ \beta \ \Gamma \ \Delta$
→ Proof $(\Gamma \ , \ [\ A \ \wedge \ \beta \] \vdash \Delta)$
→ Proof $(\Gamma \ , \ [\ \Box \ A \ \wedge \ \alpha \] \vdash \Delta)$

$\vdash \Box : \forall \{A \ \alpha \ x \ \Gamma \ \Delta\}$
→ $(\alpha \circ \langle x \rangle) \notin \text{Init} \ (\Gamma \ , \ \Delta)$
→ Proof $(\Gamma \vdash [\ A \ \wedge \ \alpha \circ \langle x \rangle \] \ , \ \Delta)$
→ Proof $(\Gamma \vdash [\ \Box \ A \ \wedge \ \alpha \] \ , \ \Delta)$

NAMOR framework: rules in Agda

$$\begin{aligned} \Diamond \vdash &: \forall \{A \alpha x \Gamma \Delta\} \\ &\rightarrow (\alpha \circ \langle x \rangle) \notin \text{Init} (\Gamma \text{ ,, } \Delta) \\ &\rightarrow \text{Proof} (\Gamma \text{ ,, } [A \hat{\wedge} \alpha \circ \langle x \rangle] \vdash \Delta) \\ &\rightarrow \text{Proof} (\Gamma \text{ ,, } [\Diamond A \hat{\wedge} \alpha] \vdash \Delta) \end{aligned}$$
$$\begin{aligned} \vdash \Diamond &: \forall \{M A \alpha \beta \Gamma \Delta\} \\ &\rightarrow \text{modalConstraint } M \alpha \beta \Gamma \Delta \\ &\rightarrow \text{Proof} (\Gamma \vdash [A \hat{\wedge} \beta] \text{ ,, } \Delta) \\ &\rightarrow \text{Proof} (\Gamma \vdash [\Diamond A \hat{\wedge} \alpha] \text{ ,, } \Delta) \end{aligned}$$

Implementation in Agda of constraints

```
modalConstraint : Logic → position → position → List pf → List pf → Type
modalConstraint S5  $\alpha$   $\beta$   $\Gamma$   $\Delta$  = Unit
modalConstraint S4dot2  $\alpha$   $\beta$   $\Gamma$   $\Delta$  =  $\alpha \subseteq \beta$ 
modalConstraint S4  $\alpha$   $\beta$   $\Gamma$   $\Delta$  =  $\alpha \sqsubseteq \beta$ 
modalConstraint T  $\alpha$   $\beta$   $\Gamma$   $\Delta$  =  $\alpha \triangleleft^0 \beta$ 
modalConstraint D  $\alpha$   $\beta$   $\Gamma$   $\Delta$  =  $\alpha \triangleleft \beta$ 
modalConstraint D4  $\alpha$   $\beta$   $\Gamma$   $\Delta$  =  $\alpha \sqsubseteq \beta \times (\Gamma \text{ ,, } \Delta) \text{ has } \beta$ 
modalConstraint K4  $\alpha$   $\beta$   $\Gamma$   $\Delta$  =  $\alpha \sqsubseteq \beta \times (\Gamma \text{ ,, } \Delta) \text{ has } \beta$ 
modalConstraint K  $\alpha$   $\beta$   $\Gamma$   $\Delta$  =  $\alpha \triangleleft \beta \times (\Gamma \text{ ,, } \Delta) \text{ has } \beta$ 
```

```
cutConstraint : Logic → mf → position → List pf → List pf → List pf → List pf → Type
cutConstraint K A  $\alpha$   $\Gamma_1$   $\Gamma_2$   $\Delta_1$   $\Delta_2$  = ( $\alpha \in \text{Init } (\Gamma_1 \text{ ,, } (\Delta_1 - A \hat{=} \alpha))$ )  $\uplus$  ( $\alpha \in \text{Init } ((\Gamma_2 - A \hat{=} \alpha) \text{ ,, } \Delta_2)$ )
cutConstraint K4 A  $\alpha$   $\Gamma_1$   $\Gamma_2$   $\Delta_1$   $\Delta_2$  = ( $\alpha \in \text{Init } (\Gamma_1 \text{ ,, } (\Delta_1 - A \hat{=} \alpha))$ )  $\uplus$  ( $\alpha \in \text{Init } ((\Gamma_2 - A \hat{=} \alpha) \text{ ,, } \Delta_2)$ )
cutConstraint _ A  $\alpha$   $\Gamma_1$   $\Gamma_2$   $\Delta_1$   $\Delta_2$  = Unit
```